Definitions: Many-valued Logics

Philosophical Logic 2025/2026

1 Many-valued logics

1.1 Syntax

Definition 1.1 (Language).

$$\phi \ ::= p \mid \neg\phi \mid \phi \vee \phi \mid \phi \wedge \phi \mid \phi \to \phi$$

1.2 Semantics

The basic building block of three-valued systems is the so-called logical matrix, which specifies

1. A finite non-empty set of truth values T
2. A set $T^+ \subseteq T$ of designated truth values
3. For each n-place connective, a truth value function $v : T^n \to T$. If $n = 0$, $v(\cdot) \in T$

Definition 1.2 (Satisfiability). *A formula ϕ is satisfiable by a valuation v iff $v(\phi) \in T^+$*

Definition 1.3 (Validity). *A formula ϕ is valid iff $v(\phi) \in T^+$ for all valuations v.*

Definition 1.4 (Entailment). *Given a set of formulas Γ and a formula ϕ, we say that Γ entails ϕ and we write $\Gamma \vDash \phi$ iff for any valuation v s.t. $v(\gamma) \in T^+$ for all $\gamma \in \Gamma$, then $v(\phi) \in T^+$.*

1.2.1 Strong Kleene K_3^s

$T^+ = \{1\}$

$\wedge$	1	i	0
1	1	i	0
i	i	i	0
0	0	0	0

$\vee$	1	i	0
1	1	1	1
i	1	i	i
0	1	i	0

$\to$	1	i	0
1	1	i	0
i	1	i	i
0	1	1	1

$\neg$	
1	0
i	i
0	1

For truth degrees $x, y \in \{0, \frac{1}{2}, 1\}$:

$$\neg x = 1{-}x, \qquad x{\wedge}y = \min(x, y), \qquad x{\vee}y = \max(x, y), \qquad x \to y = \max(1{-}x,\ y)$$

1.2.2 Weak Kleene K_3^w

$T^+ = \{1\}$

$\wedge$	1	i	0
1	1	i	0
i	i	i	i
0	0	i	0

$\vee$	1	i	0
1	1	i	1
i	i	i	i
0	1	i	0

$\to$	1	i	0
1	1	i	0
i	i	i	i
0	1	i	1

$\neg$	
1	0
i	i
0	1

1.2.3 Łukasiewicz Ł3

$T^+ = \{1\}$

$\wedge$	1	i	0
1	1	i	0
i	i	i	0
0	0	0	0

$\vee$	1	i	0
1	1	1	1
i	1	i	i
0	1	i	0

$\to$	1	i	0
1	1	i	0
i	1	1	i
0	1	1	1

$\neg$	
1	0
i	i
0	1

$\neg$, $\wedge$, and $\vee$ as in Strong Kleene.

$$x \to y = \min(1,\ 1 - x + y)$$

1.2.4 Logic of Paradox

Logic of Paradox (LP) has the same semantic clauses of K_3^s.

$T^+ = \{1, i\}$

2 Fuzzy Logic

Extend the chosen three-valued semantic clauses pointwise to the continuum of truth degrees $[0, 1]$ (e.g., $\neg x = 1 - x$, $\wedge = \min$, $\vee = \max$, and either $x \to y = \max(1 - x, y)$ or $x \to y = \min(1, 1 - x + y)$). Valuations now map atoms to $[0, 1]$ and extend compositionally.

Definition 2.1 (Truth-preserving consequence). $\Gamma \vDash_1 \varphi$ *iff for every valuation v,*

$$(\forall \gamma \in \Gamma,\ v(\gamma) = 1) \ \Rightarrow\ v(\varphi) = 1$$

Definition 2.2 (Degree-preserving consequence). $\Gamma \vDash_{\deg} \varphi$ *iff for every valuation v and every threshold $t \in [0, 1]$,*

$$(\forall \gamma \in \Gamma,\ v(\gamma) \geq t) \ \Rightarrow\ v(\varphi) \geq t$$